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De insigni usu calculi imaginariorum in calculo integrali

Leonhard Euler

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DE INSIGNI VSV
CALCVLI IMAGINARIORVM
 IN CALCULO INTEGRALI.

Auctore
L. EVLERO.

Conventui exhibuit die 3 Nov. 1777.

§. I.

Com super integrale formulae differentialis

$$\frac{\partial x (1 + x x)^2}{(1 - x x) \sqrt[4]{(1 - 6 x x + x^4)}}$$

eruissem, quod, posito brevitatis gratia $\sqrt[4]{(1 - 6 x x + x^4)} = v$,
 inveneram

$$= \frac{1}{2} l \frac{1 + x x + v v - 2 v x}{1 + x x + v v + 2 v x} - \text{Arc. tang. } \frac{2 v x}{1 + x x - v v},$$

affirmare non dubitavi hoc ipsum integrale non nisi ope
 Calculi Imaginariorum obtineri posse. Traclaveram enim ante
 istam formulam differentialem:

$$\frac{\partial y (1 - y y)^2}{(1 + y y) \sqrt[4]{(1 + 6 y y + y^4)}}$$

A 2

CX

ex qua illa oritur, si statuatur $y = x\sqrt{-1}$. Nunc ergo quoque, postquam in integrali posterioris loco y scripsissem $x\sqrt{-1}$, integrale superioris prodire debebat. Ad hoc autem requirebatur, ut tam logarithmi, quam arcus quantitatum imaginariarum ita evolverentur, ut ad formam generalem $A + B\sqrt{-1}$ reducerentur.

§. 2. Hoc autem phaenomenon in innumeris aliis casibus occurrere potest, qui ex hac consideratione originem trahunt. Sit Z eiusmodi functio ipsius z , ut formulae differentialis $Z \partial z$ integrale utcumque, sive algebraice, sive per logarithmos, sive arcus circulares exprimi queat, quod integrale per litteram V designemus, ut sit $\int Z \partial z = V$. In loco z substituamus quantitatem imaginariam quamcunque quam uti constat semper tali forma repraesentare liceat $z = y(\cos. \theta + \sqrt{-1} \sin. \theta)$, ubi angulum θ ut constante spectabimus, ita ut sola y sit variabilis; hoc modo erit $\partial z = \partial y(\cos. \theta + \sqrt{-1} \sin. \theta)$; functio autem Z recipi similem formam $Z = M + N\sqrt{-1}$, ita ut iam formula integranda sit

$\int Z \partial z = \int \partial y (M \cos. \theta - N \sin. \theta) + \sqrt{-1} \int \partial y (M \sin. \theta + N \cos. \theta)$
cuius prior pars est realis, posterior vero imaginaria.

§. 3. Fiat nunc eadem substitutio. nempe $Z = (\cos. \theta + \sqrt{-1} \sin. \theta)$ in integrali invento V , unde par forma imaginaria $P + Q\sqrt{-1}$ prodeat necesse est; et quoniam partes reales et imaginariae seorsim inter se comparari debent, hinc oriuntur duae sequentes aequalitates:

$$P = \cos. \theta \int M \partial y - \sin. \theta \int N \partial y;$$

$$Q = \sin. \theta \int M \partial y + \cos. \theta \int N \partial y;$$

unde colligimus

$$\begin{aligned} \int M \partial y &= P \cos. \theta + Q \sin. \theta \text{ et} \\ \int N \partial y &= Q \cos. \theta - P \sin. \theta \end{aligned}$$

hocque modo si inventae fuerint binae quantitates P et Q , ambo integralia tam $\int M \partial y$ quam $\int N \partial y$ exhiberi poterunt.

§. 4. Nisi autem functio proposita Z fuerit admodum simplex, plerumque litterae M et N hinc proveniunt functiones tam complicatae novae variabilis y , ut vix alia via pateat, harum formularum $\int M \partial y$ et $\int N \partial y$ integralia investigandi, praeter hanc ipsam, quam modo indicavimus, et quae per imaginaria procedit; totum ergo negotium huc redit, ut ex invento integrali V ambae quantitates P et Q inde oriundae definiantur. Quatenus igitur istud integrale V partes continet algebraicas, ista operatio nulla laborat difficultate; quando autem logarithmos et arcus circulares involvit, haud exigua sagacitate opus est, ut eius valor in formam $P + Q \sqrt{-1}$ transmutetur, quam ob rem subsidia hio sum traditurus, quibus omnes huiusmodi transformationes perfici queant.

§. 5. Cuncta autem haec subsidia commodissime repeti possunt ex sola formula $\text{Arc. tang. } t \sqrt{-1}$: Cum enim eius differentiale sit $= \frac{\partial t \sqrt{-1}}{1-t^2}$, huius integrale vicissim erit $= \frac{\sqrt{-1}}{2} \log \frac{1+t}{1-t}$, siquidem ita definiatur, ut evanescat posito $t=0$, quandoquidem hoc casu etiam arcus evanescit. Hinc igitur iam nati sumus hanc primam reductionem:

$$\text{Arc. tang. } t \sqrt{-1} = \frac{\sqrt{-1}}{2} \log \frac{1+t}{1-t}$$

ubi in generali forma $A + B \sqrt{-1}$ est $A=0$.

§. 6.

§. 6. Ponamus nunc $t = u\sqrt{-1}$, eritque
 $t\sqrt{-1} = -u$ et $\text{Arc. tang. } t\sqrt{-1} = -\text{Arc. tang. } u$,
 ex quo habebimus

$$-\text{Arc. tang. } u = \frac{\sqrt{-1}}{2} l \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}},$$

unde vicissim colligitur

$$l \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = -\frac{2}{\sqrt{-1}} \text{Arc. tang. } u = +2\sqrt{-1} \text{Arc. tang. } u.$$

Cum porro sit

$$\frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = \frac{(1+u\sqrt{-1})^2}{1+uu}, \text{ erit}$$

$$l \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = 2l(1+u\sqrt{-1}) - 2l\sqrt{-1}(1+uu)$$

$$= 2\sqrt{-1} \text{Arc. tang. } u,$$

unde colligitur haec nova redutio:

$$l(1+u\sqrt{-1}) = l\sqrt{-1}(1+uu) + \sqrt{-1} \text{Arc. tang. } u.$$

§. 7. Cum igitur omnes formulae imaginariae ad
 formam $p(\text{cof. } a + \sqrt{-1} \text{ fin. } a)$ reduci queant, erit

$$lp(\text{cof. } a + \sqrt{-1} \text{ fin. } a) = lp \text{ cof. } a + l(1 + \sqrt{-1} \text{ tang. } a)$$

et posito $u = \text{tang. } a$, fiet

$$l(1 + \text{tang. } a \sqrt{-1}) = -l \text{ cof. } a + a\sqrt{-1}.$$

Hinc deducimus istam reductionem non minus memorabilem:

$$lp(\text{cof. } a + \sqrt{-1} \text{ fin. } a) = lp + a\sqrt{-1}$$

ideoque

$$l(\text{cof. } a + \sqrt{-1} \text{ fin. } a) = a\sqrt{-1}.$$

§. 8. Hinc igitur iam facilem modum impetravimus
 omnium quantitatum imaginariarum logarithmos ad formam
 $A + B\sqrt{-1}$ revocandi. At vero pro arcubus imaginariis
 hanc

hanc solam reductionem adhuc sumus nati, qua erat Arc. tang. $t\sqrt{-1} = \frac{\sqrt{-1}}{2} \log \frac{1+t}{1-t}$. Desideratur ergo adhuc regula huiusmodi arcum imaginarium Arc. tang. $(p + q\sqrt{-1})$ ad formam $A + B\sqrt{-1}$ reducendi. Talis quidem regula iam passim reperitur, quia autem plerumque nimis operose est exuta, sequenti modo eam immediate ex solo principio hic stabilito deducemus.

§. 9. Quaeramus scilicet primo summam huiusmodi binorum arcuum, quae sit

Arc. tang. $(p + q\sqrt{-1}) + \text{Arc. tang. } (p - q\sqrt{-1})$
quam designemus littera R, et cum in genere sit

$$A \text{ tang. } a + A \text{ tang. } b = \text{Arc. tang. } \frac{a+b}{1-ab},$$

ob $a = p + q\sqrt{-1}$ et $b = p - q\sqrt{-1}$, erit

$$R = \text{Arc. tang. } \frac{2p}{1 - pp - qq}.$$

Simili modo ponatur eorundem arcuum differentia

Arc. tang. $(p + q\sqrt{-1}) - \text{Arc. tang. } (p - q\sqrt{-1}) = S,$
et quia

$$\text{Arc. tang. } a - \text{Arc. tang. } b = \text{Arc. tang. } \frac{a-b}{1+ab}, \text{ erit}$$

$$S = \text{Arc. tang. } \frac{2q\sqrt{-1}}{1 + pp + qq}.$$

Initio autem vidimus esse

$$\text{Arc. tang. } t\sqrt{-1} = \frac{\sqrt{-1}}{2} \log \frac{1+t}{1-t},$$

unde sumto $t = \frac{2q}{1 + pp + qq}$, erit

$$S = \frac{\sqrt{-1}}{2} \log \frac{(1+q)^2 + pp}{(1-q)^2 + pp}.$$

§. 10. Inventis igitur binarum illarum formularum
tam summa R quam differentia S, utramque seorsim exhibere

licet; erit enim

$\text{Arc. tang. } (p + q\sqrt{-1}) = \frac{R+S}{2}$, ideoque
 $\text{Arc. tang. } (p + q\sqrt{-1}) = \frac{1}{2} \text{Arc. tang. } \frac{2p}{1-p^2-q^2} + \frac{\sqrt{-1}}{4} \text{Arc. tang. } \frac{(1+q)^2+p}{(1-q)^2+p}$
 similique modo erit
 $\text{Arc. tang. } (p - q\sqrt{-1}) = \frac{1}{2} \text{Arc. tang. } \frac{2p}{1-p^2-q^2} - \frac{\sqrt{-1}}{4} \text{Arc. tang. } \frac{(1+q)^2+p}{(1-q)^2+p}$
 quae quidem ex priore sponte deducitur, loco q scriber
 $-q$. Hic commodè $\text{Arc. tang. } \frac{2p}{1-p^2-q^2}$ in duos resolvere lic
 quo factò erit

$$\text{Arc. tang. } (p + q\sqrt{-1}) = \frac{1}{2} \text{Arc. tang. } \frac{p}{1-q} + \frac{1}{2} \text{Arc. tang. } \frac{p}{1+q} + \frac{\sqrt{-1}}{4} \text{Arc. tang. } \frac{(1+q)^2+p}{(1-q)^2+p}$$

§. 11. Nunc igitur loco $p + q\sqrt{-1}$ substitua
 formam $r(\cos. \alpha + \sqrt{-1} \sin. \alpha)$, ut sit $p = r \cos. \alpha$ et $q = r \sin. \alpha$
 ac reperiatur

$$\text{Arc. tang. } r(\cos. \alpha + \sqrt{-1} \sin. \alpha) = \frac{1}{2} \text{Arc. tang. } \frac{2r \cos. \alpha}{1-r^2} + \frac{\sqrt{-1}}{4} \text{Arc. tang. } \frac{1+2r \sin. \alpha + r^2}{1-2r \sin. \alpha + r^2}$$

Per posteriorem autem formam erit quoque

$$\text{Arc. tang. } r(\cos. \alpha + \sqrt{-1} \sin. \alpha) = \frac{1}{2} \text{Arc. tang. } \frac{r \cos. \alpha}{1-r \sin. \alpha} + \frac{1}{2} \text{Arc. tang. } \frac{r \cos. \alpha}{1+r \sin. \alpha} + \frac{\sqrt{-1}}{4} \text{Arc. tang. } \frac{1+2r \sin. \alpha + r^2}{1-2r \sin. \alpha + r^2}$$

§. 12. Hae iam formulae hactenus inventae c
 subsidia complectuntur, quibus indigebimus ad omnes
 rithmos et arcus circulares imaginarios resolvendos. F
 las autem inventas hic simul aspectui exponamus:

$$\text{I. } l(a + b\sqrt{-1}) = la + l\left(1 + \frac{b\sqrt{-1}}{a}\right) = l\sqrt{(aa + bb)} + \sqrt{-1} \text{Arc. tang. } \frac{b}{a}$$

unde deducitur ista saepissime occurrens:

$$l \frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = 2\sqrt{-1} \text{ Arc. tang. } \frac{b}{a}.$$

Porro etiam notetur haec formula:

$$l a (\cos. \alpha + \sqrt{-1} \sin. \alpha) = l a + a \sqrt{-1}.$$

Pro arcubus autem has adepti sumus formulas:

$$\begin{aligned} \text{Arc. tang. } (a + b\sqrt{-1}) &= \frac{1}{2} \text{ Arc. tang. } \frac{2a}{1-a^2-b^2} \\ &+ \frac{\sqrt{-1}}{4} l \frac{(1+b^2)^2 + a^2}{(1-b^2+a^2)}, \end{aligned}$$

vel etiam

$$\begin{aligned} \text{Arc. tang. } a (\cos. \alpha + \sqrt{-1} \sin. \alpha) &= \frac{1}{2} \text{ Arc. tang. } \frac{2a \cos. \alpha}{1-a^2} \\ &+ \frac{\sqrt{-1}}{4} l \frac{1+2a \sin. \alpha + a^2}{1-2a \sin. \alpha + a^2}. \end{aligned}$$

§. 13. His fundamentis constitutis consideremus casus, quibus integrale $\int Z \partial z$ per logarithmos et arcus circulares exprimi potest, id quod semper evenit, quando Z est functio rationalis ipsius z , tum autem integrale componitur ex huiusmodi partibus:

- I. $l(1 \pm z)$;
- II. $l(1 - 2z \cos. \alpha + z^2)$;
- III. $\text{Arc. tang. } \frac{z \sin. \alpha}{1 - z \cos. \alpha}$;

vel saltem integralia, quae reperiuntur, facile ad tales formas redigi possunt. Harum ergo resolutionem, quando statuitur $z = y (\cos. \theta + \sqrt{-1} \sin. \theta)$, nonnullas in sequentibus problematibus expediemus.

Problema 1.

§. 14. Hanc formulam logarithmicam $l(1 \pm z)$, posito $z = y \cos. \theta + \sqrt{-1} \sin. \theta$ ad formam generalem $A + B\sqrt{-1}$ reducere.

Nova Acta Acad. Imp. Scient. Tom. XII.

B

Solu-

Solutio.

Evolvamus primo formulam

$l(1+z) = l(1+y \cos. \theta + y \sqrt{-1} \sin. \theta)$
 et comparatione cum superiore forma generali facta erit
 $a = 1 + y \cos. \theta$ et $b = y \sin. \theta$,
 unde colligitur

$$l(1+z) = l \sqrt{1 + 2y \cos. \theta + yy} + \sqrt{-1} \text{Arc. tang. } \frac{y \sin. \theta}{1 + y \cos. \theta}.$$

Hinc autem alter casus $l(1-z)$ sponte derivatur, fumendo y negative, eritque ergo $l(1-z) =$
 $l \sqrt{1 - 2y \cos. \theta + yy} - \sqrt{-1} \text{Arc. tang. } \frac{y \sin. \theta}{1 - y \cos. \theta}.$

Saepe numero autem in integralibus occurrere solet formula $l \frac{1+z}{1-z}$, cuius ergo valor, posito
 $z = y(\cos. \theta + \sqrt{-1} \sin. \theta)$,
 sequenti modo exprimetur:

$$l \frac{1+z}{1-z} = \frac{1}{2} l \frac{1 + 2y \cos. \theta + yy}{1 - 2y \cos. \theta + yy} + \sqrt{-1} \text{Arc. tang. } \frac{y \sin. \theta}{1 - y \cos. \theta},$$

quare si ambo arcus in unum contrahantur, prodibit
 $l \frac{1+z}{1-z} = \frac{1}{2} l \frac{1 + 2y \cos. \theta + yy}{1 - 2y \cos. \theta + yy} + \sqrt{-1} \text{Arc. tang. } \frac{2y \sin. \theta}{1 - yy}.$

Problema 2.

§. 15. Proposita formula logarithmica

$$l(1 - 2z \cos. \alpha + zz),$$

si in ea ponatur $z = y(\cos. \theta + \sqrt{-1} \sin. \theta)$, eius valorem ad formulam postulatam $A + B \sqrt{-1}$ reducere.

Solut

Solutio.

Si hic immediate substitutionem facere vellemus, in calculos satis molestos delaberemur, quos ut evitemus, observasse iuvabit, formulam $1 - 2z \cos. \alpha + z^2$ esse productum ex his factoribus:

$[1 - z(\cos. \alpha + \sqrt{-1} \sin. \alpha)] [1 - z(\cos. \alpha - \sqrt{-1} \sin. \alpha)]$,
quorum ergo logarithmos invicem addi oportet.

Trademus ergo primo formulam

$$l[1 - z(\cos. \alpha + \sqrt{-1} \sin. \alpha)],$$

et cum sit

$$y(\cos. \theta + \sqrt{-1} \sin. \theta)(\cos. \alpha + \sqrt{-1} \sin. \alpha) \\ = z(\cos. \alpha + \sqrt{-1} \sin. \alpha)$$

quoniam in genere est

$$(\cos. \beta + \sqrt{-1} \sin. \beta)(\cos. \gamma + \sqrt{-1} \sin. \gamma) \\ = \cos. (\beta + \gamma) + \sqrt{-1} \sin. (\beta + \gamma), \text{ erit}$$

$$l[1 - z(\cos. \alpha + \sqrt{-1} \sin. \alpha)] \\ = l[1 - y \cos. (\alpha + \theta) + \sqrt{-1} \sin. (\alpha + \theta)].$$

Hic, ergo facta comparatione erit

$$a = 1 - y \cos. (\alpha + \theta) \text{ et } b = -y \sin. (\alpha + \theta),$$

unde eius valor resolutus erit

$$l[1 - z(\cos. \alpha + \sqrt{-1} \sin. \alpha)] \\ = \frac{1}{2} l[1 - 2y \cos. (\alpha + \theta) + yy] - \sqrt{-1} \text{Arc. tang. } \frac{y \sin. (\alpha + \theta)}{1 - y \cos. (\alpha + \theta)}.$$

Hinc altera formula facile deducitur, fumendo angulum negative, eritque

$$l[1 - z(\cos. \alpha - \sqrt{-1} \sin. \alpha)] \\ = \frac{1}{2} l[1 - 2y \cos. (\theta - \alpha) + yy] - \sqrt{-1} \text{Arc. tang. } \frac{y \sin. (\theta - \alpha)}{1 - y \cos. (\theta - \alpha)}.$$

B 2

Nunc

Nunc igitur tantum opus est, ambos valores, quos modo invenimus, invicem addere, sicque prodibit hacc redutio:

$$\begin{aligned} & l(1 - 2z \cos. \alpha + z^2) \\ &= \frac{1}{2} l[1 - 2y \cos. (\alpha + \theta) + y^2] - \sqrt{-1} \text{Arc. tg. } \frac{y \sin. \alpha + \theta}{1 - y \cos. (\alpha + \theta)} \\ &+ \frac{1}{2} l[1 - 2y \cos. (\theta - \alpha) + y^2] - \sqrt{-1} \text{Arc. tg. } \frac{y \sin. (\theta - \alpha)}{1 - y \cos. (\theta - \alpha)} \end{aligned}$$

Problema 3.

§. 16. *Proposita formula pro arcu circulari*

$$T = \text{Arc. tang. } \frac{z \sin. \alpha}{1 - z \cos. \alpha},$$

si in ea ponatur $z = y(\cos. \theta + \sqrt{-1} \sin. \theta)$, eius valorem inde resultantem ad formam $A + B\sqrt{-1}$ revocare.

Solutio.

Quia hic in numeratore et denominatore imaginari occurrunt, ad simplicio rem formam perveniemus, si utriusque addamus $\text{Arc. } \alpha$, five $\text{Arc. tang. } \frac{\sin. \alpha}{\cos. \alpha}$; sic enim erit

$$T + \alpha = \text{Arc. tang. } \frac{\sin. \alpha}{\cos. \alpha - z} = 90^\circ - \text{Arc. tang. } \frac{\cos. (\alpha - z)}{\sin. \alpha}$$

ideoque

$$T = 90^\circ - \alpha - \text{Arc. tang. } \frac{\cos. (\alpha - z)}{\sin. \alpha}.$$

Iam in hac postrema formula ponamus

$$z = y(\cos. \theta + \sqrt{-1} \sin. \theta), \text{ fietque}$$

$$\text{Arc. tang. } \frac{\cos. (\alpha - z)}{\sin. \alpha} = \text{Arc. tang. } \frac{\cos. \alpha - y \cos. \theta - y \sqrt{-1} \sin. \alpha}{\sin. \alpha}$$

quae expressio comparata cum formula generali

Arc. tang. $(a + b \sqrt{-1})$ dat

$$a = \frac{\cos. \alpha - y \cos. \theta}{\sin. \alpha} \text{ et } b = -\frac{y \sin. \theta}{\sin. \alpha}.$$

Hinc igitur erit

$$1 - a a - b b = -\frac{\cos. 2\alpha + 2y \cos. \alpha \cos. \theta - y y}{\sin. \alpha^2}$$

ideoque

$$\frac{2a}{1 - a a - b b} = \frac{\sin. 2\alpha - 2y \sin. \alpha \cos. \theta}{-\cos. 2\alpha + 2y \cos. \alpha \cos. \theta - y y}, \text{ ergo}$$

$$\text{Arc. tang. } \frac{2a}{1 - a a - b b} = -\text{Arc. tang. } \frac{\sin. 2\alpha - 2y \sin. \alpha \cos. \theta}{\cos. 2\alpha - 2y \cos. \alpha \cos. \theta + y y}$$

Iam pro parte imaginaria erit

$$1 + a a + b b = \frac{1 - 2y \cos. \alpha \cos. \theta + y y}{\sin. \alpha^2},$$

unde colligitur numerator

$$(1 + b)^2 + a a = \frac{1 - 2y \cos. (\theta - \alpha) + y y}{\sin. \alpha^2}$$

et denominator

$$(1 - b)^2 + a a = \frac{1 - 2y \cos. (\alpha + \theta) + y y}{\sin. \alpha^2},$$

sicque pars imaginaria erit

$$\frac{\sqrt{-1}}{4} \sqrt{\frac{1 - b^2 - a a}{(1 - b)^2 + a a}} = \frac{\sqrt{-1}}{4} \sqrt{\frac{1 - 2y \cos. (\theta - \alpha) + y y}{1 - 2y \cos. (\alpha + \theta) + y y}},$$

quamobrem hinc colligimus

$$\text{Arc. tang. } \frac{\cos. (\theta - \alpha)}{\sin. \alpha} = -\frac{1}{2} \text{Arc. tang. } \frac{\sin. 2\alpha - 2y \sin. \alpha \cos. \theta}{\cos. 2\alpha - 2y \cos. \alpha \cos. \theta + y y}$$

$$+ \frac{\sqrt{-1}}{4} \sqrt{\frac{1 - 2y \cos. (\theta - \alpha) + y y}{1 - 2y \cos. (\alpha + \theta) + y y}}.$$

His iam formulis inventis reductio ipsius formulae propositae ita se habebit:

$$\text{Arc. tang. } \frac{2 \sin. \alpha}{1 - 2 \cos. \alpha} = 90^\circ - \alpha + \frac{1}{2} \text{Arc. tg. } \frac{\sin. 2\alpha - 2y \sin. \alpha \cos. \theta}{\cos. 2\alpha - 2y \cos. \alpha \cos. \theta + y y}$$

$$- \frac{\sqrt{-1}}{4} \sqrt{\frac{1 - 2y \cos. (\theta - \alpha) + y y}{1 - 2y \cos. (\alpha + \theta) + y y}}.$$

Hae

Hac iam reductiones haud difficulter ad omnes formulas accommodari poterunt, quod quo clarius appareat, sequens exemplum adiungamus.

Integratio
Formulae differentialis

$$\frac{\partial x}{(3 - xx) \sqrt[3]{(1 - 3xx)}} = \partial V.$$

§. 17. Quoniam nondum apparet, quomodo hanc ipsam formulam tractari conveniat, eam ad sequentem formam imaginariam, ponendo $x = z \sqrt{-1}$, reducamus, ut fit

$$\partial V = \frac{\partial z \sqrt{-1}}{(3 + zz) \sqrt[3]{(1 + 3zz)}},$$

quae forma iam ita comparata deprehenditur, ut per praecpta non ita pridem tradita ad integrationem perducere possit, eius ergo resolutionem sequenti modo expedire poterimus.

§. 18. Ponamus igitur

$$\frac{\partial z}{(3 + zz) \sqrt[3]{(1 + 3zz)}} = \partial T,$$

ut fit $V = T \sqrt{-1}$. Hanc autem formam sequenti modo repraesentemus: $\partial T = \frac{z \partial z}{(3z + z^3) \sqrt[3]{(1 + 3zz)}}$, ubi brevitas

gratia statuamus $\sqrt[3]{(1 + 3zz)} = v$, ut fit

$$\partial T = \frac{z \partial z}{v(3z + z^3)},$$

hicque secundum nostra praecepta statuamus $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, unde fit $p + q = \frac{2}{v}$ et $p - q = \frac{2z}{v}$, hincque $z = \frac{p-q}{p+q}$, ideoque differentiando

$$\partial z = \frac{2q \partial p - 2p \partial q}{(p+q)^2} = \frac{1}{2} v v (q \partial p - p \partial q),$$

quo valore substituto impetramus

$$\partial T = \frac{v z (q \partial p - p \partial q)}{2(3z + z^3)}.$$

§. 19. Cum iam fit $1 + z = pv$ et $1 - z = qv$, erit primo $2z = v(p - q)$, tum vero summa cuborum dabit

$$(1 + z)^3 + (1 - z)^3 = v^3(p^3 + q^3) = 2 + 6zz.$$

Quoniam igitur posuimus $\sqrt[3]{(1 + 3zz)} = v$, erit $v^3 = 1 + 3zz$; quam ob rem habebimus $p^3 + q^3(1 + 3zz) = 2 + 6zz$, consequenter $p^3 + q^3 = 2$. Denique vero differentia cuborum praebet $(p^3 - q^3)v^3 = 6z + 2z^3$; unde patet esse $3z + z^3 = \frac{1}{2}(p^3 - q^3)v^3$; at vero differentia quadratorum dat $(pp - qq)vv = 4z$, unde fit $z = \frac{1}{4}vv(pp - qq)$.

§. 20. Substituuntur nunc isti valores loco z et $3z + z^3$, atque nostra formula evadet

$$\partial T = \frac{(pp - qq)(q \partial p - p \partial q)}{4(1 - q^3)},$$

ubi ergo tantum binae litterae p et q occurrunt, quae ita a se invicem pendent, ut fit $p^3 + q^3 = 2$, ideoque differentiando $pp \partial p + qq \partial q = 0$, consequenter sive $\partial p = \frac{-qq \partial q}{pp}$, sive $\partial q = \frac{-pp \partial p}{qq}$.

§. 21.

§. 21. Dividatur nunc haec forma in duas partes, ponendo

$$\frac{p \cdot p (q \partial p - p \partial q)}{p^3 - q^3} = \partial P \text{ et } \frac{q \cdot q (q \partial p - p \partial q)}{p^3 - q^3} = \partial Q,$$

vt fit $\partial T = \frac{1}{4} \partial P - \frac{1}{4} \partial Q$, hicque statim patet, si in prior formula loco $p \cdot p \partial p$ scribatur $- q \cdot q \partial q$, tum prodire

$$\partial P = - \frac{\partial q (p^3 + q^3)}{p^3 - q^3}.$$

Quia vero est $p^3 + q^3 = 2$, ideoque $p^3 = 2 - q^3$, elementum ∂P per solam litteram q ita exprimetur, ut fit

$$\partial P = \frac{\partial q}{1 - q^3}.$$

§. 22. Simili modo si in altera formula ∂Q loco $q \cdot q \partial q$ scribatur $- p \cdot p \partial p$, prodibit $\partial Q = \frac{\partial p (p^3 + q^3)}{p^3 - q^3}$, quae ergo ob relationem inter p et q suppeditat hanc formulam:

$$\partial Q = \frac{\partial p}{p^3 - 1} = \frac{-\partial p}{1 - p}.$$

His igitur coniundis erit

$$4 \partial T = \frac{\partial p}{1 - p^3} - \frac{\partial q}{1 - q^3},$$

ficque totum negotium perductum est ad duas formulas differentiales rationales, quas ergo per logarithmos et arcus circulares integrare licet.

§. 23. Ad haec integralia invenienda statuatur

$$\frac{1}{1 - p^3} = \frac{F}{1 - p} + \frac{G}{1 + p + p^2},$$

ubi notetur fore

$$F = \frac{1 - p}{1 - p^3} = \frac{1}{1 - p + p^2},$$

posito $1 - p = 0$, sive $p = 1$, unde fit $F = \frac{1}{3}$, tum vero erit

$$G = \frac{1 + p - p^2}{1 - p^3} = \frac{1}{1 - p},$$

po-

posito $1 + p + pp = 0$. Hic iam ut littera p ex denomina-
tore tolli queat, multiplicetur supra et infra per $2 + p$, fiet
 $G = \frac{p}{2 - p - pp}$. Quia igitur est $p + pp = -1$, erit $G = \frac{2+p}{3}$;
quam ob rem habebimus $\frac{3 \partial p}{1 - p^3} = \frac{\partial p}{1 - p} + \frac{(2 + p) \partial p}{1 + p + pp}$. Con-
stat autem esse $\int \frac{\partial p}{1 - p} = -l(1 - p)$ et

$$\int \frac{p \partial p + 2 \partial p}{p p + p + 1} = \frac{1}{2} l(1 + p + pp) + \frac{3}{2} \int \frac{\partial p}{1 + p + pp}.$$

Novimus autem in genere esse

$$\int \frac{\partial p}{1 - 2p \cos. \alpha + p^2} = \frac{1}{\sin. \alpha} \text{Arc. tang. } \frac{p \sin. \alpha}{1 - p \cos. \alpha},$$

unde patet sumi debere $\alpha = 120^\circ$, et ob $\sin. \alpha = \frac{\sqrt{3}}{2}$, erit

$$\int \frac{\partial p}{1 - p - p^2} = \frac{2}{\sqrt{3}} \text{Arc. tang. } \frac{p \sqrt{3}}{2 + p},$$

ficque totum integrale erit

$$3 \int \frac{\partial p}{1 - p^3} = -l(1 - p) + \frac{1}{2} l(1 + p + pp) \\ + \sqrt{3} \text{Arc. tang. } \frac{p \sqrt{3}}{2 + p},$$

similique modo erit

$$3 \int \frac{\partial q}{1 - q^3} = -l(1 - q) + \frac{1}{2} l(1 + q + qq) \\ + \sqrt{3} \text{Arc. tang. } \frac{q \sqrt{3}}{3 + q}.$$

§. 24. His igitur inventis erit

$$12 T = l \frac{1 - q}{1 - p} + \frac{1}{2} l \frac{1 + p + pp}{1 + q + qq} \\ + \sqrt{3} \text{Arc. tang. } \frac{p \sqrt{3}}{2 + p} - \sqrt{3} \text{Arc. tang. } \frac{q \sqrt{3}}{2 + q}.$$

Quare cum fit $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, habebimus

$$T = \frac{1}{12} l \frac{v - 1 + z}{v - 1 - z} + \frac{1}{24} l \frac{v v + v(1+z) + (1+z)^2}{v v + v(1-z) + (1-z)^2} \\ + \frac{1}{4\sqrt{3}} \text{Arc. tang. } \frac{(1+z)\sqrt{3}}{2v + 1 + z} - \frac{1}{4\sqrt{3}} \text{Arc. tang. } \frac{(1-z)\sqrt{3}}{2v + 1 - z}.$$

§. 25. Nunc secundum praecepta supra exposita, ubi sumimus $z = y(\cos. \theta + \sqrt{-1} \sin. \theta)$, quia est $x = z\sqrt{-1}$, erit

$$z = -x\sqrt{-1} = y(\cos. \theta + \sqrt{-1} \sin. \theta);$$

unde patet statui debere $\theta = 90^\circ$ et $y = -x$, hocque notato, ut superiores reductiones ad nostrum casum propius accommodemus, ibi ubique loco z et y scribamus $\frac{z}{s}$ et $\frac{y}{s}$, quo facto reductiones erunt

$$\text{I. } l(s+z) = +\frac{1}{2}l(ss+2sy\cos.\theta+yy) + \sqrt{-1} \text{Arc.tg. } \frac{y\sin.\theta}{s+y\cos.\theta}.$$

$$\text{II. } l(ss-2sz\cos.\alpha+zz) = \frac{1}{2}l[ss-2sy\cos.(\alpha+\theta)+yy] \\ - \sqrt{-1} \text{Arc.tang. } \frac{y\sin.(\alpha+\theta)}{s-y\cos.(\alpha+\theta)} \\ + \frac{1}{2}l(ss-2sy\cos.(\theta-\alpha)+yy) - \sqrt{-1} \text{Arc.tang. } \frac{y\sin.(\theta-\alpha)}{s-y\cos.(\theta-\alpha)}.$$

$$\text{III. } \text{Arc.tang. } \frac{z\sin.\alpha}{s-z\cos.\alpha} = 90^\circ - \alpha + \frac{1}{9} \text{Arc.tg. } \frac{ss\sin.2\alpha-2sy\sin.\alpha\cos.\theta}{ss\cos.2\alpha-2sy\cos.\alpha\cos.\theta+yy} \\ - \frac{\sqrt{-1}}{4} l \frac{ss-2sy\cos.(\theta-\alpha)+yy}{ss-2sy\cos.(\alpha+\theta)+yy}.$$

§. 26. Iam haec praecepta ad singulas partes integralis inventi applicemus, ac primo quidem pro formula $l(v-1+z)$ erit $s = v-1$, et ob $y = -x$ et $\theta = 90^\circ$ colligitur

$$\text{I. } l(v-1+z) = \frac{1}{2}l[(v-1)^2+xx] - \sqrt{-1} \text{Arc.tang. } \frac{x}{v-1}.$$

$$\text{II. } l(v-1-z) = \frac{1}{2}l[(v-1)^2+xx] + \sqrt{-1} \text{Arc.tang. } \frac{x}{v-1}.$$

III. Pro formula $l[vv+v(1+z)+(1+z)^2]$ patet fore

$$ss = vv + v + 1, \text{ seu } s = \sqrt{vv+v+1};$$

$$\cos. \alpha = \frac{v-2}{2\sqrt{vv+v+1}} \text{ et } \sin. \alpha = \frac{v\sqrt{3}}{2\sqrt{vv+v+1}},$$

unde ob $\theta = 90^\circ$ erit

$$\cos. (\alpha + \theta) = -\sin. \alpha; \cos. (\theta - \alpha) = \sin. \alpha;$$

$$\sin. (\alpha + \theta) = \cos. \alpha \text{ et } \sin. (\theta - \alpha) = \cos. \alpha;$$

quo

quo observato erit:

$$\begin{aligned} l[vv + v(1+z) + (1+z)^2] &= \frac{1}{2}l[vv + v + 1 - vx\sqrt{3} + xx] \\ &+ \frac{1}{2}l[vv + v + 1 + vx\sqrt{3} + xx] \\ &- \sqrt{-1} \text{Arc. tang. } \frac{x(v+2)}{2(vv+v+1) - vx\sqrt{3}} \\ &- \sqrt{-1} \text{Arc. tang. } \frac{x(v+2)}{2(vv+v+1) + vx\sqrt{3}} \end{aligned}$$

hinc simul mutato signo litterarum, z et x erit

$$\begin{aligned} l[vv + v(1-z) + (1-z)^2] &= \frac{1}{2}l[vv + v + 1 + vx\sqrt{3} + xx] \\ &+ \frac{1}{2}l[vv + v + 1 - vx\sqrt{3} + xx] \\ &+ \sqrt{-1} \text{Arc. tang. } \frac{x(v+2)}{2(vv+v+1) + vx\sqrt{3}} \\ &+ \sqrt{-1} \text{Arc. tang. } \frac{x(v+2)}{2(vv+v+1) - vx\sqrt{3}} \end{aligned}$$

§. 27. Nunc porro pro Arc. tang. $\frac{(1+z)\sqrt{3}}{2v+1+z}$, quae forma in regula nostra non continetur, notetur esse

Arc. tg. $\frac{(1+z)\sqrt{3}}{2v+1+z} = \text{Arc. tg. } \frac{\sqrt{3}}{2v+1} + \text{Arc. tg. } \frac{vz\sqrt{3}}{2(vv+v+1) + (v+2)z}$,
qui postremus valor comparatus cum Arc. tang. $\frac{z \sin. \alpha}{1 - z \cos. \alpha}$ iterum praebet $\sin. \alpha = \frac{v\sqrt{3}}{2\sqrt{(vv+v+1)}}$, sicque anguli $\alpha + \theta$ et $\theta - \alpha$ manent iidem, ut ante; unde redutio praebet

$$\begin{aligned} \text{Arc. tang. } \frac{vz\sqrt{3}}{2(vv+v+1) + (v+2)z} &= -\frac{1}{2} \text{Arc. tg. } \frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xz} \\ &- \frac{\sqrt{-1}}{4} l \frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx} \end{aligned}$$

Nadi ergo sumus has reductiones:

$$\begin{aligned} \text{Arc. tg. } \frac{(1+z)\sqrt{3}}{2v+1+z} &= \text{Arc. tg. } \frac{\sqrt{3}}{2v+1} - \frac{1}{2} \text{Arc. tg. } \frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xz} \\ &- \frac{\sqrt{-1}}{4} l \frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx} \text{ et} \end{aligned}$$

$$\begin{aligned} \text{Arc. tg. } \frac{(1-z)\sqrt{3}}{2v+1-z} &= \text{Arc. tg. } \frac{\sqrt{3}}{2v+1} - \frac{1}{2} \text{Arc. tg. } \frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xz} \\ &- \frac{\sqrt{-1}}{4} l \frac{vv+v+1-vx\sqrt{3}+xx}{vv+v+1+vx\sqrt{3}+xx} \end{aligned}$$

§. 28. Quodsi iam omnes has partes rite colligamus, reperiemus

$$T = -\frac{\sqrt{-1}}{6} \text{Arc. tang. } \frac{x}{v-1} - \frac{\sqrt{-1}}{12} \text{Arc. tg. } \frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}} \\ - \frac{\sqrt{-1}}{12} \text{Arc. tg. } \frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}} - \frac{\sqrt{-1}}{8\sqrt{3}} \int \frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$

§. 29. Hic igitur commode usu venit, ut omnes partes reales se mutuo destruxerint, imaginariae vero duplicatae prodierint, quemadmodum natura rei manifesto postulat. Cum igitur integrale quaesitum sit $V = T\sqrt{-1}$, nunc eius valor pulcherrime prodit realis, quocirca perducti sumus ad hanc integrationem:

$$\frac{\partial x}{(3-xx)\sqrt[3]{(1-3xx)}} = +\frac{1}{6} \text{Arc. tang. } \frac{x}{v-1} \\ + \frac{1}{12} \text{Arc. tang. } \frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}} \\ + \frac{1}{12} \text{Arc. tang. } \frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}} \\ + \frac{1}{8\sqrt{3}} \int \frac{(vv+v+1+vx\sqrt{3}+xx)}{vv+v+1-vx\sqrt{3}+xx},$$

ubi est $v = \sqrt[3]{(1-3xx)}$. Hae formulae aliquanto simpliciores reddi possunt, considerando quod sit $1-v^3=3xx$ ideoque $1+v+vv = \frac{3xx}{1-v}$, unde cum plures substitutiones adhiberi queant, iis hic non immorandum censemus, se contenti esse possumus, istius formulae differentialis integratu cruisse, ad quod per nullam aliam methodum aditus potest videri.

§. 30. Caeterum calculus facilius evadet, si in integrali primum invento ambo arcus per $\frac{1}{4\sqrt{3}}$ multiplicati in unum colligantur: inde enim prodit $\frac{1}{4\sqrt{3}} \text{Arc.tg.} \frac{vz\sqrt{3}}{vv+v+1-zx}$. Hic iam statim ponatur $z = -x\sqrt{-1}$, ut formula prodeat $-\frac{1}{4\sqrt{3}} \text{Arc.tang.} \frac{vx\sqrt{3}\sqrt{-1}}{vv+v+1+xx}$, quae comparata cum canonica $\text{Arc.tang.} t\sqrt{-1} = \frac{\sqrt{-1}}{2} \log \frac{1+t}{1-t}$, ob $t = \frac{vx\sqrt{3}}{vv+v+1+xx}$ statim perducit ad hanc formulam reduciam:

$$-\frac{\sqrt{-1}}{8\sqrt{3}} \log \frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$